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ON LINEAR TOPOLOGICAL PROPERTIES OF H^1 ON SPACES OF HOMOGENEOUS TYPE

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ABSTRACT. Let (X, d, μ) be a space of homogeneous type. Let $B = \{x \in X: \mu\{x\} = 0\}$, then $\mu(B) > 0$ implies that $H^1(X, d, \mu)$ contains a complemented copy of $H^1(\delta)$. This applies to Hardy spaces $H^1(\partial\Omega, d, \omega)$ associated to weak solutions of uniformly elliptic operators in divergence form. Under smoothness assumptions of the coefficients of the elliptic operators, we obtain that $H^1(\partial\Omega, d, \omega)$ is isomorphic to $H^1(\delta)$.

INTRODUCTION

The motivation for this work was the investigation of linear topological properties of Hardy spaces $H^1(\partial\Omega, d, \omega)$ associated to weak solutions of uniformly elliptic operators in divergence form

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{i,j}(x)\frac{\partial u}{\partial x_j}(x)\right) = 0$$

where $a_{i,j}(x)$ are bounded real measurable functions on a Lipschitz domain Ω (cf. [J-K, F-J-K, C-F-M-S]).

In particular we are interested in the question whether these Hardy spaces are isomorphic to the dyadic $H^1(\delta)$ space (cf. [Ma]).

As usual one breaks up this problem into two:

Problem A. Does $H^1(\partial\Omega, d, \omega)$ contain a complemented copy of $H^1(\delta)$?

Problem B. Does $H^1(\delta)$ contain a complemented copy of $H^1(\partial\Omega, d, \omega)$?

By Pelczynski's decomposition method, a positive solution to A and B implies that $H^1(\partial\Omega, d, \omega)$ is isomorphic to $H^1(\delta)$.

The positive solution to Problem A is obtained as a Corollary to our Theorem 1.4. Theorem 1.4 applies also to $H^1_{\text{at}}(S_n)$ (cf. [W]). Hence it gives Wojtaszczyk's result that $H^1(\delta)$ is isomorphic to a complemented subspace of $H^1_{\text{at}}(S_n)$ without using Alexandrov's result on inner functions in B_n.

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(For $n = 1$ the above-mentioned result is due to Maurey. There E. M. Stein's multiplier theorem for $H^1(D)$ functions is used in an essential way (cf. [Ma, §2].)

The solution of Problem B is obtained in the following way. First we show that $H^1(\partial\Omega, d, \omega)$ is isomorphic to a certain space $H^1_{\text{prob}}(\Omega, \omega)$ of continuous martingales.

Then, by Lemma 2.16 and Proposition 2.13, the probabilistic methods of Maurey (cf. [Ma, §§3 and 4]) (see also Wolniewicz [W]) allows us to show that $H^1_{\text{prob}}(\Omega, \omega)$ is isomorphic to a complemented subspace of $H^1(\delta)$.

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Definition 1.1. Let (X, d, μ) be a space of homogeneous type [C-W, p. 587]. A function $a \in L^1(X, \mu)$ is an atom for (X, d, μ) if $\int a\, d\mu = 0$ and if there exists a ball $I \subset X$ such that $\operatorname{supp} a \subset I$ and $\|a\|_\infty \leq 1/\mu(I)$.

Remark. For any homogeneous space (X, d, μ) there exists a quasimetric m on $X \times X$, equivalent to d, such that (X, d, μ) is a normal homogeneous space of order α, and such that the atoms for (X, d, μ) are the same as for (X, m, μ). More precisely, there exists $C > 0$ such that each $a \in L^1(X, \mu)$ is an atom for (X, d, μ) iff $C \cdot a$ is an atom for (X, m, μ) (cf. [M-S-2, pp. 272, 273]). Normal homogeneous spaces of order α are studied in detail in [M-S-1, M-S-2]. From now on we will work with (X, m, μ),

Definition 1.2a. Fix $f \in L^2(X, \mu)$. Let

$$\|f\|_{H^1(X, m, \mu)} = \inf\Big\{\sum |\lambda_i|\colon \text{there exists a sequence of atoms } a_j \text{ for } (X, m, \mu) \text{ such that } f = \sum \lambda_i a_i\Big\}.$$

If no such sequence exists, we let $\|f\|_{H^1(X, m, \mu)} = \infty$. Then

$$H^1(X, m, \mu) := \{f \in L^1(X, \mu)\colon \|f\|_{H^1(X, m, \mu)} < \infty\}.$$

Remark. $H^1(X, d, \mu)$ and $H^1(X, m, \mu)$ are identical (cf. [M-S-2]).

Definition 1.2b. Let ϕ be a function on X; let f be in $L^1(X, \mu)$. Let $\overline{\alpha} = \frac{\alpha}{2}$. Then

$$L(\phi, \overline{\alpha}, m) := \sup\left\{\frac{|\phi(x) - \phi(y)|}{m(x, y)^{\overline{\alpha}}}\colon x, y \in X\right\},$$

$$f_s(x) := \operatorname{supp}\left\{\frac{1}{s}\int_X f(y)\phi(y)\, d\mu(y)\colon s \geq 0,\ \text{and}\ \operatorname{supp}\phi \subset B(x, s), L(\phi, \overline{\alpha}, m) \leq s^{-\overline{\alpha}}, \|\phi\|_\infty \leq 1\right\}$$

where $B(x, s) = \{y \in X, m(x, y) < s\}$.

$$f^*(x) := \sup_{s>0} |f_s(x)|.$$

The main result in [M-S-2] gives

Theorem 1.3. *There exists $C > 0$ such that for every $f \in L^1(X, \mu)$ we have*

$$\frac{1}{C}||f||_{H^1(X,m,\mu)} \leq ||f^*||_{L^1(X)} \leq C||f||_{H^1(X,m,\mu)}.$$

For later purposes we single out a special subset of X namely: $B = \{x \in X: \mu(\{x\}) = 0\}$.

The result to be proved in this section is as follows:

Theorem 1.4. *If $\mu(B) > 0$ then $H^1(X, m, \mu)$ contains a complemented subspace Y which is isomorphic to the dyadic $H^1(\delta)$.*

Example 1.5. Let μ be the counting measure on $\mathbf{Z}$, $m(x, y) = |x - z|$, $x, y \in \mathbf{Z}$. Then $(\mathbf{Z}, m, \mu)$ is a normal homogeneous space of order 1 which does not satisfy $\mu(B) > 0$.

Examples which satisfy $\mu(B) > 0$ will be discussed in §2.

Remark. Our strategy is as follows: We want to construct a system of functions $f_{ni}: X \to \mathbf{R}$, which is equivalent to the Haar system in $H^1(\delta)$. Then, exploiting the fact that f_{ni} can be chosen (almost) biorthogonal, we will use the orthogonal projection to show that $\operatorname{span}\{f_{ni}\}$ is isomorphic to a complemented subspace of $H^1(X, m, \mu)$.

We use the assumption $\mu(B) > 0$ to show that in X there exists a "fat" collection $\mathscr{G}$ of balls, which behave (practically) like a subcollection of dyadic intervals. Inside each ball in $\mathscr{G}$ we will find two collections of disjoint balls, $\mathscr{E}_1, \mathscr{E}_2$ ($\subseteq \mathscr{G}$) which cover two disjoint sets of (almost) the same measure (Lemma 1.9). Moreover for each ball $K \subset \mathscr{G}$ we will find a function a_K which serves as a good substitute for the characteristic function of K. (a_K can be written as the sum of functions with a known Lipschitz constant (Lemma 1.12)).

Lemmas 1.9 and 1.12 allow us to define functions g_1, g_2 such that $g_1 - g_2$ behaves like a Haar function (Lemma 1.13). A crucial consequence of our construction is that $|g_1 - g_2|$ and $(g_1 - g_2)^*$ have (essentially) the same support.

Lemma 1.6. *Let (X, m, μ) be a homogeneous space. Let $B = \{x \in X: \mu(\{x\}) = 0\}$. If $\mu(B) > 0$ then there exists a family $\mathscr{G}$ of balls in X such that:*

(6a) *$I, J \in \mathscr{G}$, $I \cap J \neq \emptyset$ implies $I \subset J$ or $J \subset I$.*

(6b) *$I \subset J$ implies $\mu(I) \leq \mu(J)/2$.*

(6c) *$\mu(\{t \in X:$ t lies in infinitely many $I \in \mathscr{G}\}) > 0$.*

Proof. First we observe that B coincides with $\bigcap_n \bigcup_{x \in B} B(x, 2^{-n})$. Suppose not, then there exists a sequence $\{x_m\} \in B$ such that $\lim x_n = x$ and $x \notin B$. That means $\mu(\{x\}) > 0$. In particular $x_n \neq x$ for all x_n. On the other hand, by [M-S-1, Theorem 1] there exists $r_x > 0$ such that $B(x, r_x) \setminus \{x\} = \varnothing$—a contradiction. The construction of $\mathscr{G}$ will be a consequence of the following:

Sublemma. *For each ball $I \subset X$ and $\eta > 0$ there exists a finite collection $\mathscr{C} = \{C_i\}$ of pairwise disjoint balls such that*

(a) $$C_i \subset I, \quad \mu(C_i) \le \mu(I)/2.$$

(b) $$\mu((B \cap I) \setminus \cup C_i) < 2\eta\mu(B \cap I).$$

Proof of the Sublemma. Fix $\frac{1}{2} > \varepsilon > 0$ depending on η and X.

Step 0. For $x \in I \cap B$, we have $0 = \mu(\{x\}) = \lim \mu(B(x, 2^{-n}))$. Hence for each $x \in I \cap B$ there exists a ball $B_x \subset I$ with center x such that $\mu(B_x) \le \mu(I)/2$. Let $\mathscr{E}_0 := \{B_x : x \in I \cap B\}$. $\mathscr{E}_0$ is an open covering of $I \cap B$.

By the Vitali Wiener covering lemma [C-W, p. 623] there exists K, depending only on (X, m, μ), and a pairwise disjoint sequence $\mathscr{D}_0 = \{D_m\}$ of balls in $\mathscr{E}_0$ such that

$$\bigcup (4 \cdot K) D_n \supset B \cap I.$$

Here $(4 \cdot K)D_n$ denotes the ball with the same center as D_n but with radius $4K$ times bigger than that of D_n. This implies

$$\sum_{n=1}^{\infty} \mu(D_n) \ge \frac{1}{C^{4K}} \sum \mu((4K) \cdot (D_n)) \ge \frac{1}{C^{4K}} \mu(B \cap I).$$

Here C denotes the doubling constant of the homogeneous space (X, m, μ). Finally we choose $n_1 \in \mathscr{N}$ such that

$$\sum_{n=1}^{n_1} \mu(D_n) > \frac{1}{2C^4 K} \mu(B \cap I).$$

Next we choose $C_n \subset D_n$, balls of a slightly smaller radius such that $\mu(D_n \setminus C_n) \le \frac{\varepsilon}{2}\mu(D_n)$. In particular

$$\mu(\overline{C}_n \setminus C_n) \le \frac{\varepsilon}{2} \cdot \mu(D_n).$$

$C_1, \ldots, C_{n_1}$ are the first n_1 elements of $\mathscr{C}$.

To prepare our next step, we let $E_1 := \{x \in B \cap I : x \notin \bigcup \overline{C}_n\}$, $\widetilde{E}_1 := \{x \in B \cap I : x \notin \bigcup C_n\}$. First $\mu(\widetilde{E}_1 \setminus E_1) \le \frac{\varepsilon}{2}\mu(I \cap B)$. Second

$$\begin{aligned} \mu(E_1) \le \mu(\widetilde{E}_1) &\le \mu(I \cap B) - \sum \mu(C_n) \\ &\le \mu(I \cap B) - \left(1 - \frac{\varepsilon}{2}\right) \sum \mu(D_n) \\ &\le \mu(I \cap B)\left(1 - \left(1 - \frac{\varepsilon}{2}\right)\frac{1}{2C^{4K}}\right). \end{aligned}$$

Step 1. For $x \in E_1$, there exists a ball $B_x \subset E_1$, $x \in B_x$, such that $\mu(B_x) < \mu(I)/2$. This implies in particular that $B_x \cap (\bigcup C_n) = \varnothing$. Let

$$\mathscr{E}_1 = \{B_x : x \in E_1, B_x \subset E_1, \mu(B_x) \le \mu(I)/2\}.$$

$\mathscr{E}_1$ is an open covering of E_1. Repeating the argument of Step 0 we obtain a collection $\mathscr{D}_1 = \{D_n^1\}$ of pairwise disjoint balls such that for n_2 large enough:

$$\sum_{n=1}^{n_2} \mu(D_n^1) \geq \mu(E_1)\frac{1}{2C^{4K}}.$$

Next choose $C_n^1 \subset D_n^1$ such that $\mu(D_n^1 \setminus C_n^1) \leq \varepsilon\mu(D_n^1)$. Then $C_{n_1+j} := C_j^1$, $j \leq n_2$ are the next n_2 elements of $\mathscr{C}$. Again for

$$E_2 := \left\{x \in E_1 : x \notin \bigcup_{n_1}^{n_2} \overline{C}_j\right\}, \quad \widetilde{E}_2 := \left\{x \in E_1 : x \notin \bigcup_{n_1}^{n_2} C_j\right\}$$

we have

$$\mu(\widetilde{E}_2 \setminus E_2) \leq \varepsilon\mu(E_1),$$
$$\mu(E_2) \leq \mu(\widetilde{E}_2) \leq \mu(E_1)\left(1 - (1-\varepsilon)\frac{1}{2C^{4K}}\right).$$

After Step p we have constructed pairwise disjoint balls $C_1, \dots, C_{n_1+\cdots+n_p}$, and sets $E_1, \dots, E_{p+1}$; $\widetilde{E}_1, \dots, \widetilde{E}_{p+1}$ such that for $1 \leq j \leq p$

(1) $\widetilde{E}_j \subset E_{j-1} \subset \widetilde{E}_{j-1}$,
(2) $\mu(\widetilde{E}_j \setminus E_j) \leq \varepsilon\mu(E_j)$,
(3) $\mu(\widetilde{E}_j) \leq \mu(B \cap I)(1 - 1/4C^{4K})^j$,
(4) $(I \cap B) \setminus \bigcup_{i=1}^{n_1+\cdots+n_j} C_i = (\widetilde{E}_1 \setminus E_1) \cup \cdots \cup (\widetilde{E}_j \setminus E_j) \cup \widetilde{E}_j$.

((3) follows from the assumption $0 < \varepsilon < \frac{1}{2}$.)

Let $\delta = 1/4C^{4K}$. Then (1) and (4) imply that

$$\mu\left((I \cap B) \setminus \bigcup_{i=1}^{n_1+\cdots+n_j} C_i\right) \leq \left(\sum_{i=1}^{j} \varepsilon(1-\delta)^{i-1} + (1-\delta)^{j+1}\right)\mu(I \cap B).$$

Finally, we put $\varepsilon = \eta \cdot \frac{\delta}{2}$ and for p big enough the result follows. Here ends the proof of the sublemma.

Now we proceed as follows: Choose a ball I such that $\mu(B \cap I) > 0$ and put $G_1 = \{I\}$. Suppose we have already constructed $G_1, \dots, G_{p-1}$, then we first choose $\varepsilon_J > 0$, $J \in G_{p-1}$ such that $\sum \varepsilon_J < 4^{-p}$. Fix $J \in G_{p-1}$ and apply the sublemma to ε_J and $(J \cap B)$. We denote the resulting family by $G_1(J)$ and put $G_p = \bigcup_{J \in G_{p-1}} G_1(J)$. Moreover, we have

$$\mu(B \cap G_{p-1} \setminus G_p) \leq \mu(B \cap I)4^{-p}.$$

Finally $\mathscr{G} = \bigcup_p G_p$ satisfies (6a), (6b), and (6c).

Notation 1.7. Let $E \subset X$, $\mathscr{E} \subset \mathscr{G}$, $I \in \mathscr{G}$ be given. Then we denote

$$E \cap \mathscr{E} = \{J \in \mathscr{E} : J \subset E\}, \quad \mathscr{E}^* = \bigcup_{J \in \mathscr{E}} J,$$
$$\sigma(\mathscr{E}) = \{t \in X : t \text{ lies in infinitely many } K \in \mathscr{E}\}.$$

Later, we reserve the letter σ for $\sigma(\mathscr{G})$.

$$G_1(I) = \{J \subseteq I : J \in \mathscr{G}, J \text{ maximal}\}, \quad G_n(I) = \bigcup_{J \in G_{n-1}(I)} G_1(J).$$

For each I we have $G_n^*(I) \supset G_{n+1}^*(I)$ and $\sigma \cap I = \bigcap G_n^*(I)$. We say that $I \in \mathscr{G}$ lies below $\mathscr{E}$ iff

(a) $I \subset \mathscr{E}^*$,

(b) for each $J \in \mathscr{E}$ with $J \cap I \neq 0$ we have $J \supset I$,

(c) if $I' \in \mathscr{G}$ satisfies (a), (b) and $I' \cap I \neq \varnothing$ then $I' \subset I$.

For sets $F, G \subset X$ we let

$$m(F, G) := \inf\{m(x, y) : x \in F, y \in G\}.$$

Lemma 1.8. *Let K be a ball in (X, m, μ), $\varepsilon > 0$. Let I $(\subset K)$ be a ball in X with center x_0 and radius r. Then there exists a ball $J \subset I$, $\tau > 0$ such that $\mu(I \setminus J) < \varepsilon$ and $m(J, K \cap \complement I) > \tau$.*

Proof. First by inner regularity there exists $s < r$ such that

$$|\mu(J(x_0, s)) - \mu(I)| < \varepsilon.$$

Next fix $z_1 \in J$, $z_2 \in K \cap \complement I$. We have $m(x_0, z_1) = s - \delta$ and $m(x_0, z_2) = r + \eta$ for some positive η, δ^+. Then, invoking that X is of order α:

$$\begin{aligned}(\operatorname{diam} K)^{1-\alpha} m(z_1, z_2)^\alpha &> |m(z_1, x_0) - m(z_2, x_0)| \\ &= |s - \delta - (r + \eta)| > |s - r|.\end{aligned}$$

Hence $\tau := (|s - r| / \operatorname{diam} K^{1-\alpha})^{1/\alpha}$ is the right choice.

Lemma 1.9. *Let K be a ball in X with $\mu(K \cap \sigma) \neq 0$. For $\varepsilon > 0$ there exist $\tau > 0$, finite collections of pairwise disjoint balls $\mathscr{E}_j \subset \mathscr{G}$ such that*

$$m(\mathscr{E}_1^*, \mathscr{E}_2^*) > \tau, \tag{9a}$$

$$m(\mathscr{E}_j^*, \complement K) > \tau, \tag{9b}$$

$$\mu(\mathscr{E}_1^* \cup \mathscr{E}_2^*) \geq (1 - \varepsilon)\mu(K \cap \sigma), \tag{9c}$$

$$|\mu(\mathscr{E}_1^*) - \mu(\mathscr{E}_2^*)| \leq \varepsilon\mu(K \cap \sigma). \tag{9d}$$

Proof. By Lebesgue's theorem on differentiation, there exists $I_j \in \mathscr{G} \cap K$, pairwise disjoint such that

$$\frac{\mu(I_j \cap \sigma)}{\mu(I_j)} \geq (1 - \varepsilon), \quad \mu\left(\bigcup I_j \setminus K \cap \sigma\right) \leq \varepsilon.$$

Let $\mathscr{J}$ be a finite subcollection of $\{I_j\}$ such that $\mu(\mathscr{J}^*) > (1 - \varepsilon)\mu(\bigcup I_j)$. Fix $I \in \mathscr{J}$. For large n there exist finite disjoint $\mathscr{D}_1(I)$, $\mathscr{D}_2(I) \subset G_n(I)$ such that

$$|\mu(\mathscr{D}_j^*(I)) - \mu(I \cap \sigma)\tfrac{1}{2}| \leq \varepsilon\mu(I), \qquad j \in \{1, 2\}.$$

Next choose $K \in \mathscr{D}_j(I)$. There exists a ball $K' \subset K$ with $m(K', \complement K) > 0$, $\mu(K \setminus K') < \varepsilon\mu(K)$. Next we choose $n' \in \mathbf{N}$ large enough and obtain for $K' \cap G_{n'} := \mathscr{E}(K)$, the following estimate, $|\mu(K' \cap \sigma) - \mu(\mathscr{E}^*(K))| \leq \varepsilon\mu(K)$.

Finally, we put

$$\mathscr{E}_1 := \bigcup_{I \in \mathscr{I}} \bigcup_{K \in \mathscr{D}_1(I)} \mathscr{E}(K) \qquad \mathscr{E}_2 := \bigcup_{I \in \mathscr{I}} \bigcup_{K \in \mathscr{D}_2(I)} \mathscr{E}(K).$$

Taking into account that $\mathscr{I}$, $\mathscr{D}_j(I)$ are finite families, we are done.

Remark. We know now how to construct a "tree of sets" in X and we wish to associate "Haar functions" to this tree. The most obvious choice would be to take consecutive differences of characteristic functions as "Haar functions" (cf. [Mü, §2]). However, for technical reasons, we have to introduce certain approximations of characteristic functions. This approximation procedure is explained in Lemmas 1.12 and 1.13.

Definition 1.10. Let I be a ball in X with radius r and center x_0. Then f_I is defined as follows:

$$f_I(x) = g(x)\frac{\mu(I)}{\int g(x)\,d\mu}$$

where

$$g(x) = \begin{cases} 1 & \text{if } m(x, x_0) \leq \frac{r}{2}, \\ 2 - \frac{2m(x, x_0)}{r} & \text{if } \frac{r}{2} \leq m(x, x_0) \leq r, \\ 0 & \text{if } m(X, x_0) \geq r. \end{cases}$$

Remark 1.11. There exist $C > 0$ and $\eta > 1$ such that for each I

$$\|f_I\|_\infty \leq C \ \mu(\{x \in I: f_I < \tfrac{1}{2}\}) < \eta\mu(I), \qquad \int f_I(x)\,d\mu = \mu(I).$$

Moreover, we have the following: $x, y \in I$ implies

$$|f_I(x) - f_I(y)| \leq 2\left(\frac{m(x, y)}{r}\right)^\alpha.$$

Indeed,

$$\begin{aligned} |f_I(x) - f_I(y)| &\leq 2|(m(x, x_0) - m(y, x_0))| \\ &\leq 2m(x, y)^\alpha \cdot r^{1-\alpha} \leq 2\left(\frac{m(x, y)}{r}\right)^\alpha. \end{aligned}$$

Lemma 1.12. *For $\varepsilon > 0$, $K \in \mathscr{G}$, there exist $m_1 < m_2 \in \mathbf{N}$, $\mathscr{K} \subset G_{m_1}(K) \cup \cdots \cup G_{m_2}(K)$ such that for $\alpha_K = \sum\{f_I : I \in \mathscr{K}\}$ we have*

(12a) $$\mu(\{x \in K: a_K(x) < \tfrac{1}{2}\} \cap \sigma) \leq \varepsilon\mu(K \cap \sigma),$$

(12b) $$\mu(K \cap \sigma) = \int a_K\,d\mu,$$

(12c) $$\|a_K\|_\infty \leq 3 \cdot C.$$

Proof.

Step 0. Choose $n_0 \in \mathbf{N}$ such that $\mu(G^*_{n_0}(K) \setminus \sigma) < \frac{\varepsilon}{4}$. Put

$$g_0(x) = \sum\{f_L(x): L \in G_{n_0}(K)\}.$$

Taking into account that $L, L' \in G_{n_0}(K)$ implies $L \cap L' = \varnothing$ we see that $\mu(\{g_0 < \frac{1}{2}\} \cap \sigma) \leq \eta\mu(K \cap \sigma)$.

Step 1. $E_0 := \{g_0 < \frac{1}{2}\} \cap K$. Next choose $n_1 > n_0$ such that

$$\mu(G^*_{n_1}(E_0) \setminus \sigma) \leq \varepsilon/4^2,$$
$$g_1 = \sum\{f_L: L \in G_{n_1}(E_0)\},$$
$$E_1 := \{x \in g_0(x) + g_1(x) \leq \tfrac{1}{2}\} \cap K$$

and again

$$\begin{aligned}\mu(E_1 \cap \sigma) &\leq \mu(\{x \in E_0: g_1 \leq \tfrac{1}{2}\} \cap \sigma) \\ &\leq \eta\mu(E_0 \cap \sigma) \leq \eta^2\mu(K \cap \sigma).\end{aligned}$$

Step k. Choose $n_k > n_{k-1}$ such that

$$\mu(G_{n_k}(E_{k-1}) \setminus \sigma) \leq \varepsilon/4_{k+1}.$$

Let $g_k = \sum\{f_L: L \in G_{n_k}(E_{k-1})\}$. Next, let $E_k = \{x: (g_0 + \cdots + g_k)(x) < \frac{1}{2}\} \cap K$. Then,

$$\begin{aligned}\mu(E_k \cap \sigma) &\leq \mu(\{x \in E_{k-1}: g_k < \tfrac{1}{2}\} \cap \sigma) \\ &\leq \eta\mu(E_{k-1} \cap \sigma) \leq \eta^{k+1}\mu(K \cap \sigma).\end{aligned}$$

Let $k \in \mathbf{N}$ be big enough and put $f = \sum_{i=0}^{k} g_i$. Then

$$a_K(x) := \frac{f(x)}{\int f(x)\,d\mu}\mu(K \cap \sigma)$$

is the right choice.

Lemma 1.13. *For $I \in \mathscr{G}$, $\varepsilon > 0$, there exists $\tau_0 > 0$, $\tilde{l} \in \mathbf{N}$ such that for $l > \tilde{l}$ there exist collections of balls $\mathscr{E}_j \subset \bigcup_{k=l}^{\infty} G_k(I)$, $j \in \{0, 1\}$, positive real numbers c_K, $K \in \mathscr{E}_j$ with $1 \leq c_K < C$ such that for $g_j := \sum\{f_K c_K: K \in \mathscr{E}_j\}$ the following holds*:

(13a) $$\mu(\mathscr{E}_1^* \cup \mathscr{E}_0^*) \geq (1 - \varepsilon)\mu(I \cap \sigma),$$

(13b) $$m(\mathscr{E}_1^*, \mathscr{E}_0^*) \geq \tau_0,$$

(13c) $$m(\mathscr{E}_j^*, \complement I) \geq \tau_0, \qquad j \in \{0, 1\},$$

(13d) $$\left|\int g_0 - \int g_1\right| \leq \varepsilon\mu(I),$$

(13e) $$\mu(\{|g_1 + g_2| < \tfrac{1}{2}\} \cap \sigma) \leq \mu(I \cap \sigma)\varepsilon,$$

(13f) $$\|g_j\|_\infty \leq C, \qquad j \in \{0, 1\}.$$

Proof. First apply Lemma 1.9 to obtain finite collections of pairwise disjoint balls $\mathscr{D}_1, \mathscr{D}_2 \subset \mathscr{G}$ and which satisfy (9a), (9b), (9c), (9d). Next fix $\tilde{l} \in \mathbf{N}$, such that $J \in \mathscr{D}_j$, $K \in G_{\tilde{l}}$, $I \cap K \neq \emptyset$, imply $J \supset K$. To $l > \tilde{l}$, $K \in \mathscr{D}_j^* \cap G_l$ and $\varepsilon > 0$ we apply Lemma 1.12 to obtain $a_K(x)$. Summing up we obtain

$$g_j(x) = \sum \{\alpha_K(x) : K \in \mathscr{D}_j^* \cap G_l\}$$

which satisfies (13d), (13e), (13f). $\mathscr{E}_j := \mathscr{D}_j^* \cap G_l$ satisfies (13a), (13b), (13c).

Suppose f can be written as a sum of functions like $(g_1 - g_2)$. Then the behavior of its maximal function f^* can easily be analyzed.

Lemma 1.14. *Let $\{I_i\}$ be a sequence of pairwise disjoint balls with center $\{x_l\}$. Let $\{f_i\}$ be a sequence of continuous functions such that $|f_i| \leq C \cdot \chi_{Ii}$, and*

$$\left| \int_{I_i} f_i \, d\mu \right| \leq \varepsilon_i \mu(I_i).$$

Then for $f = \sum f_i$ we have the following estimate

$$(f)_r(x) \leq \mu\left(\bigcup I_j \right) \sup\{(\operatorname{diam} I_j)^{\overline{\alpha}} r^{\overline{\alpha}-1} : m(I_j, x) < 2r\}$$
$$+ \left(\sum \varepsilon_i \right) \cdot \sup\{\mu(I_j) \cdot r^{-1} : m(I_j, x) 2r\}.$$

Proof. To estimate $(f)_r(x)$ we choose ϕ such that $\operatorname{supp} \phi \subset B(x, r)$, $L(\phi, \overline{\alpha}, m) \leq r^{-\overline{\alpha}}$, $\|\phi\|_\infty < 1$. Then

$$\int f\phi \leq \sum_j \int_{I_j} f_j(\phi - \phi(x_j)) + \int_{I_j} f_j \phi(x_j)$$
$$\leq \sum_{\{j : m(I_j, x) < 2r\}} \operatorname{diam}(I_j)^{\overline{\alpha}}, r^{-\overline{\alpha}} \cdot \mu(I_j) \cdot C$$
$$+ \left(\sum \varepsilon_i \right) \cdot \sup\{\mu(I_j) : m(I_j, x) \leq 2r\}.$$

Remark. (a) Let $m(x, \bigcup I_j) = \tau > 0$. Let f be as above, and assume $\mu(\bigcup I_j) \leq 1$. Then $\sup_j \mu(I_j) < \delta$, and $r > r_0$ implies

$$(f)_r(x) \leq \left(1 + \sum \varepsilon_j \right) \cdot \max \left\{ \frac{\delta^{\overline{\alpha}}}{r_0} \frac{1}{r_0}, \frac{\delta}{r_0} \right\}.$$

(b) $r < \frac{\tau}{2}$ implies $(f)_r(x) = 0$.

Lemma 1.15. *Let $\mathscr{G}$ be a family of balls in X which satisfies* (6a)–(6c) *of Lemma* 1.6. *Then there exists $C \in \mathbf{R}^+$ such that for any sequence $\varepsilon_n > 0$ there exist*:

(i) *finite collections of balls $\{\mathscr{E}_{ni}\}$, $n \in \mathbf{N}$, $0 \leq i \leq 2^n - 1$.*

(ii) *real numbers $c_K \in \mathbf{R}^+$, $K \in \mathscr{E}_{ni}$ with $c_K \leq c$, such that for*

$$f_{n,i} := \sum \{c_K f_K : K \in \mathscr{E}_{n+1, 2i}\} - \sum \{c_K f_K : K \in \mathscr{E}_{n+1, 2i+1}\}$$

the following holds:

(15.1) $f_{n,i}$ *is supported on a subset of* $\mathscr{E}^*_{ni}$; $\quad \|f_{ni}\|_\infty \le C$.

(15.2) $$\mu(\{|f_{ni}| < \tfrac{1}{2}\} \cap \sigma) < \mu(\mathscr{E}^*_{ni} \cap \sigma) \cdot \varepsilon_n.$$

(15.3) (a) *for each* $r > 0$, $x \notin \mathscr{E}^*_{n,i}$ *implies* $(f_{n,i})_r(x) \le \varepsilon_n$,
(b) *there exists* $t_n \downarrow 0$ *such that for* $r_0 := 1$, $r_n := t_{n-1}/2$, *and* $x \in X$ *the following holds:* $(f_{n,i})_r(x) \le \varepsilon_n$ *for* $r \ge r_n$, $\operatorname{var}\{f_{n,i}(y) : y \in B(x,t)\} \le \varepsilon_n$, $t < t_n$.

(15.4) $$\left| \int f_{m,i} f_{n,j} \right| \le \min\{\varepsilon_n, \varepsilon_m\}.$$

(15.5) (a) $2^{-n}/C \le \mu(\mathscr{E}^*_{n,i}) \le 2^{-n}C$,
(b) $\mathscr{E}^*_{n,2i} \cup \mathscr{E}^*_{n+1,2i+1} \subset \mathscr{E}^*_{n,i}$ *and* $\mathscr{E}^*_{n,i} \cap \mathscr{E}^*_{n,j} = \varnothing$ *for* $i \ne j$.

(15.6) $L \in \mathscr{E}_{n,i}$, $K \in \mathscr{E}_{m,j}$, $L \subset K$ *implies* $m \le n$.

(15.7) $L \in \mathscr{E}_{m,j}$ *and* $m \le n$, *implies* $\mu(L \cap \mathscr{E}^*_{n,i}) \le \mu(L) \cdot 2^{-n} \cdot 2^m \cdot C$.

Proof.

Step 0. Choose $I \in \mathscr{G}$ with $\mu(I \cap \sigma(\mathscr{G})) \ge \mu(I)/2$ and $\mu(I) \le 1$, $r_0 := 1$. Then for $\varepsilon_0 > 0$ there exists $r_0 > 0$, $\tilde{l} \in \mathbf{N}$ such that for $l \ge \tilde{l}$ there exist

(i) finite collections of balls

$$\mathscr{E}_{1,j}(I,l) \subset \bigcup_{k=l}^{\infty} G_k(I), \qquad j \in \{0,1\}.$$

(ii) real numbers $c_K \in \mathbf{R}^+$, $K \in \mathscr{E}_{1,j}(I,l)$ with $c_K < c$ such that for

$$g_{1,j}(l) := \sum \{f_K c_K : K \in \mathscr{E}_{1,j}(I,l)\}, \qquad j \in \{0,1\},$$

the following holds:

(a) $$\mu(\mathscr{E}^*_{1,0}(I,l) \cup \mathscr{E}^*_{1,1}(I,l)) > (1-\varepsilon)\mu(I \cap \sigma),$$

(b) $$m(\mathscr{E}^*_{1,0}(I,l), \quad \mathscr{E}^*_{1,1}(I,l)) \ge \tau_0,$$

(c) $$m(\mathscr{E}^*_{1,j}(I,l), \complement I) \ge \tau_0,$$

(d) $$\|g_{1,j}(l)\|_\infty \le C,$$

(e) $$\left| \int g_{1,0}(l) - \int g_{1,1}(l) \right| \le \varepsilon_0 \mu(I),$$

(f) $$\mu(\{g_{1,0}(l) + g_{1,1}(l) < \tfrac{1}{2}\} \cap \sigma) \le \mu(I \cap \sigma) \cdot \varepsilon_0.$$

Finally, we choose $l_0 \in \mathbf{N}$ such that

$$\left(\frac{2^{-l_0 \alpha}}{r_0} \right) \frac{1}{r_0} + \frac{2^{-l_0}}{r_0} < \varepsilon_0$$

and let

$$f_{00} := g_{1,0}(l_0) - g_{1,1}(l_0), \quad \mathscr{E}_{1,j} := \mathscr{E}_{1,j}(I, l_0), \qquad j \in \{0, 1\}.$$

For $f_{0,0}$ there exists $t_0 > 0$ such that for $t < t_0$, $x \in X$:

$$\operatorname{var}\{f_{0,0}(y), y \in B(x, t)\} < \varepsilon_0.$$

Step n. We are given $\mathscr{E}_{n,i}$, $t_{n-1} > 0$, $\varepsilon_n > 0$ and $r_n := t_{n-1}/2$.

Fix $J \in \mathscr{G}$ below $\mathscr{E}_{ni}$. By Lemma 1.13 there exist $\tau(J) > 0$ and $\tilde{l} \in \mathbf{N}$ such that for $l > \tilde{l}$ we find:

(i) finite collections of balls $\mathscr{E}_{n+1,2i+j}(J, l)$ contained in $\bigcup_{k=l}^{\infty} G_k(J)$.

(ii) real numbers $c_K \in \mathbf{R}^+$, $K \in \mathscr{E}_{n+1,2i+j}(J, l)$ such that for

$$g_j(J, l) = \sum\{f_K c_K : K \in \mathscr{E}_{n+1,2i+j}(J, l)\}$$

and

$$A_j := \mathscr{E}^*_{n+1,2i+j}(J, l)$$

the following holds:

(a) $$\mu(A_0 \cup A_1) > (1 - \varepsilon_n)\mu(J \cap \sigma),$$

(b) $$m(A_0, A_1) > \tau(J),$$

(c) $$m(A_j, \complement J)) > \tau(J),$$

(d) $$\|g_j(J, l)\|_\infty \le C,$$

(e) $$\left| \int g_0(J, l) - \int g_1(J, l) \right| \le \varepsilon_n \mu(J),$$

(f) $$\mu(\{g_1(J, l) + g_2(J, l)) < \tfrac{1}{2}\} \cap \sigma) \le \tfrac{1}{8^n} \mu(J \cap \sigma).$$

Finally, we choose $l(J)$ such that

$$\max\{2^{-l(J)\overline{\alpha}} \cdot r_n^{-\overline{\alpha}+1}, 2^{-l(J)\overline{\alpha}} \cdot \tau(J)^{-\overline{\alpha}+1}, 2^{-l(J)} \cdot \tau(J)\} < \varepsilon_n.$$

We execute this construction for every J below $\mathscr{E}_{n,i}$. Then we put

$$\tau_n = \min\{\tau(J) : J \text{ below } \mathscr{E}_{n,i}\}, \quad l_n = \max\{l(J) : J \text{ below } \mathscr{E}_{n,i}\}.$$

We let

$$f_{n,i} := \sum\{g_0(J, l_n) - g_1(J, l_n) : J \text{ below } \mathscr{E}_{n,i}\},$$
$$\mathscr{E}_{n+1,2i+1} := \bigcup\{\mathscr{E}_{n+1,2i+j}(J, l_n) : J \text{ below } \mathscr{E}_{ni}\}.$$

One should remark that the first component in $\max(\ ,\ )$ which defines $l(J)$ is needed to ensure (15.3)(b), whereas the second and third component take care of (15.3)(a). We will use them to obtain the majorization

$$\int \sup_r (f)_r \, d\mu \le C \left\| \sum a_{mi} h_{mi} \right\|_{H^1(\delta)}.$$

Finally, we choose $t_n < t_{n-1}$ such that for $x \in X$

$$\operatorname{var}\{f_{n,i}(y) : y \in B(x,t)\} < \varepsilon_n, \qquad t \in t_n.$$

Verification of (15.1)–(15.7): Except for point (3) everything is clear: Fix $f_{n,i}$: Let $\{I_j\}$ denote the balls in $\mathscr{G}$ which are $\mathscr{E}_{n,i}$; then $f_{n,i}$ has the following representation: $f_{n,i} = \sum f_k$ where f_i is supported on a subset of I_j and

$$m(\operatorname{supp} f_k, \complement I_k) > \tau_n, \quad \int f_k = \varepsilon_k \mu(I_k), \quad \|f_k\|_\infty \le C.$$

Hence $x \notin \mathscr{E}^*_{n,i}$ implies $m(x, \bigcup I_j) > \tau_n$. By Lemma 1.14 we have

$$(f_{n,i})_r(x) = 0 \quad \text{for } r < \tau_n/2, \qquad (f_{n,i})_r(x) \le \varepsilon_n \quad \text{for } r > \tau_n/2.$$

(This follows from our choice of l_n in Step n.) 3(a) is thus verified. 3(b) follows again by the choice of l_n and the estimates in Lemma 1.14.

Proposition 1.16. *Let* $\{\mathscr{E}_{ni,}\}$ *and* $\{f_{n,i}\}$ *satisfy conditions* (15.1) *to* (15.7) *of Lemma* 1.15. *Then* $\operatorname{span}\{f_{n,i}\}$ *in* $H^1(X)$ *is isomorphic to a complemented subspace Y of* $H^1(X)$, *where* Y *is isomorphic to* $H^1(\delta)$.

Proof. Given a finite linear combination $f = \sum a_{m,j} f_{m,j}$ we have to show that there exists $C > 0$ such that

$$\frac{1}{C}\left\|\sum a_{m,j} h_{m,j}\right\|_{H^1(\delta)} \le \|f\|_{H^1(X)} \le C\left\|\sum a_{m,j} h_{m,h}\right\|_{H^1(\delta)}.$$

We start with the right-hand inequality.

Case 1. $x \notin \bigcup_n \bigcup_i \mathscr{E}^*_{ni}$ then by (15.3)(a) for each $r > 0$,

$$(f)_r(x) \le \left(\sum 2^m \varepsilon_m\right) \sup |a_{m,j}|.$$

Case 2. $x \in \bigcup_n \bigcup_i \mathscr{E}^*_{ni}$. There exists (m,j) such that $x \in \mathscr{E}^*_{(m,j)}$. If $x \notin \bigcap_n \bigcup \mathscr{E}^*_{ni}$ then there exists a minimal dyadic interval (m_0, j_0) such that $x \in \mathscr{E}_{(m_0,j_0)}$. Fix $r > 0$, and choose $n \in \mathbf{N}$ such that $r_{n+1} \le r < r_n$. Then by (15.3)

$$(f)_r(x) \le \left|\sum_{m=1}^{\max(n,m_0)} a_{m,j} f_{m,j}\right|(x) + \left(\sum_m \varepsilon_m 2^m\right) \sup |a_{m,j}| \cdot \chi \mathscr{E}^*_{00}$$

if $x \in \bigcap_n \bigcup \mathscr{E}^*_{ni}$. Then we simply get

$$(f)_r(x) \le \left|\sum_{m=1}^{n} a_{m,j} f_{m,j}\right|(x) + \left(\sum_m \varepsilon_m 2^m\right) \sup |a_{m,j}| \chi \mathscr{E}^*_{00}$$

The left-hand inequality $\sup_r (f)_r(x) \ge \sup_n \sup_{r_n < t < t_n} (f)_r(x)$ by (15.3)(b) this expression is bigger than

$$\sup_n \left|\sum_{m=0}^{n} a_{m,j} f_{m,j}(x)\right| - \left(\sum \varepsilon_m 2^m\right) \sup |a_{m,j}| \chi \mathscr{E}^*_{00}(x).$$

Hence we obtain the minorization: (using (15.2), (15.5)(a), (15.5)(b))

$$\int \sup_r (f)_r(x) \geq \left\| \sum a_{m,j} h_{m,j} \right\|_{H^1} \left(1 - \sum \varepsilon_m 2^m\right) \cdot C.$$

It remains to check that $\operatorname{span}\{f_{n,i}\}$ is isomorphic to a *complemented* subspace of $H^1(X)$. We do this by verifying the following

Claim 17.

$$\begin{array}{cccc} g: BMO(\delta) & \to & BMO(X) \\ h_{n,i} & \to & f_{n,i} \end{array}$$

is bounded.

Proof of Claim 17. Fix $\{a_{ni,}\} \in \mathbf{R}$ and let $f = \sum a_{n,i} f_{n,i}$. By construction, this sum may be written as $f = \sum b_K (f_K \cdot c_K)$ where c_K is given by Lemma 1.13. Let $\mathscr{E} := \bigcup \mathscr{E}_{n,i}$. For $K \in \mathscr{E}$ we let $\mathscr{E}(K) = (n, i)$ iff $K \in \mathscr{E}_{n,i}$. Hence b_K equals $a_{\mathscr{E}(K)}$. Fix a ball $I \subset X$ and let

$$E_1 := \{K \in \mathscr{E} : \operatorname{diam}(K) > \operatorname{diam}(I) \text{ and } K \cap I \neq \varnothing\},$$
$$E_2 := \{K \in \mathscr{E} : \operatorname{diam}(K) \leq \operatorname{diam}(I) \text{ and } K \cap I \neq \varnothing\}.$$

Let $\mathscr{J}$ denote the maximal subsets of E_2. It is clear that $\mu(\mathscr{J}^*) \leq C \cdot \mu(I)$ and $E_2 = \mathscr{E} \cap \mathscr{J}^*$. For $k \in \mathbf{N}_0$ we also introduce:

$$E_1^k := \{L \in E_1 : \operatorname{diam}(I) 2^k \leq \operatorname{diam}(L) \leq \operatorname{diam}(I) 2^{k+1}\}.$$

Independent of k, the cardinality of E_1^k is bounded by a constant only depending on X (cf. [C-W, p. 624]. This uses also that $\mathscr{E} \subset \mathscr{G}$ and the fact that X is normal).

Finally, we let

$$f_j = \sum \{b_K (f_K \cdot c_K) : K \in E_j\}; \qquad j \in \{1, 2\},$$

and let x_0 be the center of I, $x \in I$. Now we estimate (using Remark 1.10)

$$\begin{aligned} |f_1(x) - f_1(x_0)| &\leq \sum_{K \in E_1} c_K |b_K| \, |f_K(x) - f_K(x_0)| \\ &\leq c \cdot \sum_k \sum_{K \in E_1^k} |b_K| \left(\frac{m(x, x_0)}{\operatorname{diam}(I) 2^k} \right)^\alpha \\ &\leq C_\alpha \sup |b_K| = C_\alpha \sup |a_{ni}|. \end{aligned}$$

Hence

$$\left(\frac{1}{\mu(I)} \int_I (f_1(x) - f_1(x_0))^2 \, d\mu \right)^{1/2} \leq \frac{\mu(I)}{\mu(I)} C_\alpha \sup |b_K|.$$

Next we consider f_2:

$$\begin{aligned} \left(\int_I (f_2(x))^2 \right)^{1/2} &\leq \left(\int_{\mathscr{J}^*} \left(\sum_{L \in \mathscr{J}} \sum_{K \subset L} c_K \cdot f_K b_K \right)^2 \right)^{1/2} \\ &\leq \left(\sum_{L \in \mathscr{J}} \int_L \left(\sum_{\{(m,j) : (m,j) \subseteq \mathscr{E}(L)\}} f_{m,j} a_{m,j} \right)^2 \right)^{1/2} + \left(\sum_{L \in \mathscr{J}} \int_L f^2_{\mathscr{E}(L)} a^2_{\mathscr{E}(L)} \right)^{1/2}. \end{aligned}$$

This inequality holds by 15.6

$$\leq \left(\sum_{L \in \mathcal{J}} \sum_{\{(m,j):(m,j)\subset \mathcal{E}(L)\}} \int_L f_{m,j}^2 a_{m,j}^2 \right)^{1/2} 2$$

$$\leq \left(\sum_{L \in \mathcal{J}} \sum_{\{(m,j):(m,j)\subset \mathcal{E}(L)\}} \mu(L \cap \mathcal{E}_{m,j}^*) a_{m,j}^2 \right)^{1/2} 2.$$

Let $\mathcal{E}(L) = (n_L, i_L)$. Then this last expression may be estimated using 15.7) by

$$\left(\sum_{L \in \mathcal{J}} \sum_{\{(m,j)\subset (n_L, i_L)\}} \mu(L) \cdot 2^{-m} 2^{+n_L} a_{m,j}^2 \right)^{1/2}$$
$$\leq c \cdot \mu(\mathcal{J}^*)^{1/2} \left\| \sum a_{m,j} h_{m,j} \right\|_{BMO(\delta)}$$
$$\leq (\mu(I)^{1/2}) \left\| \sum a_{m,j} h_{m,j} \right\|_{BMO(\delta)} \cdot c.$$

This proves the claim (cf. [J, Lemma 1.1]).

Consider the operators

$$P: H^1(X) \to H^1(X) \quad f \to \sum \langle f, f_{ni} \rangle f_{ni} / \|f_{ni}\|_2^2.$$
$$i: H^1(\delta) \to H^1(X) \quad h_{n,i} \to f_{n,i}.$$

By the first part of the proof, $\|i\| \, \|i^{-1}\| \leq C^2$. Note that P is bounded iff $\|(i^{-1}P)^*\|_{BMO(X)}$ is bounded. But $(i^{-1}P)^*$ coincides with j. Hence $\|P\| \leq \|j\| \cdot C^2$. Unfortunately P is *not* idempotent. On the other hand by 15.4 for $f \in H^1(X)$ the following holds

$$\|PPf - Pf\| \leq \sum_{m \neq n} \min\{\varepsilon_n, \varepsilon_m\} \cdot 2^m \cdot 2^n \|f\|.$$

By a standard perturbation argument, this is enough to conclude that $i(H^1(\delta))$ is isomorphic to a complemented subspace of $H^1(X)$, provided ε_m are chosen small enough.

2

Here we describe a class of Hardy spaces to which Theorem 1.4 applies. Our description will be rather brief and the reader is assumed to be familiar with the references [P, J-K, C-F-M-S, and Ma (or W)]. We let $\Omega \subseteq \mathbf{R}^n$ be a bounded Lipschitz domain, star-shaped with respect to 0. Let

$$L = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right)$$

be a uniformly elliptic operator with bounded real measurable coefficients (i.e. there exists $\lambda \in \mathbf{R}^+$ such that for $x \in \mathbf{R}^n$ and $y \in \Omega$

$$\frac{1}{\lambda}|x|^2 < \sum_{i,j=1}^{n} a_{ij}(y)x_i x_j < \lambda|x|^2 .$$

A function u in Ω is called L-harmonic iff u is a weak solution of $Lu = 0$. The general reference to elliptic operators in divergence form is [G-T, Chapter 8] and [C-F-M-S]). We let ω^x denote L-harmonic measure for Ω evaluated at $x \in \Omega$. (ω^0 will be denoted by ω.)

For $y \in \partial\Omega$, $\lambda > 0$, $B(y,r) := \{z \in \mathbf{R}^n : |y - z| < r\}$ and $\Delta(y,r) := B(y,r) \cap \partial\Omega$. In [C-F-M-S] it was established that L-harmonic measure on a Lipschitz domain Ω together with Euclidean metric on $\partial\Omega$ form a space of homogeneous type. More precisely they proved

Theorem 2.1. *There exists $\eta \in \mathbf{R}^+$ such that for all $y \in \partial\Omega$, $r > 0$ we have*

$$\omega(\Delta(y,r)) > \eta\omega(\Delta(y,2r)).$$

We may thus define the atomic Hardy space $H^1(\partial\Omega, d, \omega)$ as in §1.

Theorem 2.2. *The atomic Hardy space $H^1(\partial\Omega, d, \omega)$ contains a complemented copy of dyadic $H^1(\delta)$.*

Proof: We wish to apply Theorem 1.4. It is enough to check that for each $x \in \partial\Omega$: $\omega(\{x\}) = 0$. Suppose to the contrary that there exists $x_0 \in \partial\Omega$ with $\omega(\{x_0\}) > 0$. Now one constructs $y_n \in \partial\Omega$, $r_n \in \mathbf{R}^+$ such that for $m \neq n$, $\Delta(y_n, r_n) \cap \Delta(y_m, r_m) = 0$ and $x_0 \in \Delta(y_n, 2r_n)$. By the doubling property there exists $\eta > 0$ such that for each n we have $\omega(\Delta(y_n, r_n)) > \eta\omega(\{x_0\})$ which contradicts the fact that ω is a finite measure.

We now turn to the converse questions:

Is $H^1(\partial\Omega, d, \omega)$ isomorphic to a complemented subspace of dyadic $H^1(\delta)$? To answer this question I was forced to search for different representations (or descriptions) of $H^1(\partial\Omega, d, \omega)$. It will be shown here that (under continuity assumptions of $((\partial/\partial x_i)a_{ij}(x)))$, $H^1(\partial\Omega, d, \omega)$ can be identified with a certain Hardy space $H^1_{\text{prob}}(\Omega, \omega)$ associated to diffisions in Ω.

Finally, the probabilistic methods of B. Maurey [Ma] which were developed further by T. Wolniewicz [W], can be used to show that $H^1_{\text{prob}}(\Omega, \omega)$ contains a complemented copy of $H^1(\delta)$.

The above-mentioned identification is done in two steps. We first consider

Hardy spaces associated to maximal functions. Fix $Q \in \partial\Omega$. Then $\Gamma(Q) = \{x \in \Omega : |x - Q| < 2\operatorname{dist}(x, \partial\Omega)\}$. We let u be defined in Ω; then $Nu(Q) := \sup\{u(x) : x \in \Gamma(Q)\}$. Now we define $H^1(\Omega, \omega) := \{u : Lu = 0$ and $Nu \in L^1(\partial\Omega, \omega)\}$. In Theorem 8.13 of [J-K], D. Jerison and C. Kenig showed in particular that the Banach spaces $H^1(\Omega, \omega)$ and $H^1(\partial\Omega, d, \omega)$ are isomorphic. Moreover, an explicit and natural isomorphism is given. As a consequence of this isomorphism the following description of the dual space of $H^1(\Omega,\omega)$ is obtained.

Let $\Delta = \Delta(x, r)$ for some $x \in \partial\Omega$, $r > 0$. Let f be locally integrable with respect to ω and put

$$f_\Delta = \frac{1}{w(\Delta)} \int_\Delta f(x)\, dw(x).$$

f is said to belong to $BMO(\partial\Omega, \omega)$ iff

$$\sup \frac{1}{w(\Delta)} \int_\Delta (f - f_\Delta)^2\, dw < \infty$$

where sup is extended over all "balls" $\Delta \subset \partial\Omega$. Then we have

Theorem 2.3 [J-K, p. 25]. *There exists $C > 0$ such that for functions u, v on $\partial\Omega$ we have*

$$\int_{\partial\Omega} u(y)v(y)\, dw \le C\|u\|_{H^1(\Omega,\omega)} \cdot \|v\|_{BMO(\partial\Omega,\omega)}.$$

$$H^1(\Omega,\omega)^* \cong BMO(\partial\Omega,\omega).$$

Hardy spaces associated to diffusions in Ω. We let $(Y_t, \mathscr{F}_t, \sum, \mathbf{P}^x)$ be a strong Markov process with almost sure continuous trajectories, and with infinitesimal generator extending L. Moreover, we demand that for any regular subdomain $\Omega' \subseteq \Omega$ and any bounded measurable function $f: \partial\Omega' \to \mathbf{R}$ we have

$$\int_\Sigma f(Y_{\tau_{\Omega'}}\, d\mathbf{P}^x = \int_{\partial\Omega'} f(z)\, dw^x_{\Omega'}(z)$$

where $\tau_{\Omega'} := \inf\{t : Y_t \notin \Omega'\}$.

Finally, we assume that for each $x \in \Omega$, $\mathbf{P}^x\{\tau_\Omega < \infty\} = 1$. It is well known that we may obtain such a Markov process provided the functions a_{ij} are two times continuously differentiable (see [Øk, Dy, II]). Subsequently we denote $\mathbf{P}^0$ by $\mathbf{P}$ and τ_Ω by τ. Now we are prepared to state

Definition 2.4. Let u be a function defined on Ω. Then

$$u^* := \sup_{t>0} |u(Y_{\tau\wedge t})|,$$

$$H^1_{\text{prob}}(\Omega,\omega) = \left\{u : \Omega \to \mathbf{R} : Lu = 0 \text{ and } \int_\Sigma u^* d\mathbf{P} > \infty\right\}$$

and we let

$$\|u\|_{H^1_{\text{prob}}(\Omega,\omega)} := \int_\Sigma u^* d\mathbf{P}.$$

$H^1_{\text{prob}}(\Omega,\omega)$ will be (naturally) identified with $H^1(\Omega,\omega)$. Again, for our purposes *any* isomorphism between these two spaces would have been enough.

Theorem 2.5. *There exists $C > 0$ such that for any L-harmonic function u the following holds*:

$$\frac{1}{C}\|u\|_{H^1_{\text{prob}}(\Omega,\omega)} \le \|u\|_{H^1(\Omega,\omega)} \le C \cdot \|u\|_{H^1_{\text{prob}}(\Omega,\omega)}.$$

Remark. This theorem is an extension of a result of Burkholder, Gundy, Silverstein (see [P, p. 36]). All ingredients of the proof given below are already contained in the literature. The left-hand inequality in Theorem (2.5) may be treated in the same manner as the right-hand inequality in [P, Theorem 4]. But instead of elementary properties of the Poisson kernel for the unit disk, we have to use the following result.

Theorem 2.6 ([C-F-M-S], see also [D-J-K]). *There exists* $\eta > 0$ *such that for each* $x \in \Omega$ *and for* $y_0 \in \partial\Omega$ *satisfying* $|x - y_0| = \operatorname{dist}(x, \partial\Omega) =: s$ *we obtain* $w^x(\Delta(y_0, s)) > \eta$.

If we use Theorem 2.6 properly, the left-hand side of Theorem 2.5 may be proved as Proposition 2 in [W] (cf. also [P, pp. 37–40]). We only have to observe

Lemma 2.7. *Let* $A \subset \partial\Omega$ *be closed. Let* $B \subset \Omega$ *be the sawtooth region above* A *(see* [D-J-K, *p.* 99]). *Then for each* $x \in \Omega \setminus B$, $w^x(\complement A) > \delta$ *(where* δ *is independent of* A, B *or* x).

Proof. Fix $x \in \Omega \setminus B$. Let

$$E(x) = \{y \in \partial\Omega : |y - x| < 2\operatorname{dist}(x, \partial\Omega)\}.$$

$E(x) \cap \complement A$ contains now a ball with radius comparable to $\operatorname{dist}(x, \partial\Omega)$ and center y_0 satisfying $|x - y_0| = \operatorname{dist}(x, \partial\Omega)$. Hence, by Theorem 2.6 and Theorem 2.7, we get

$$w^x(\complement A) \geq w^x(\complement a \cap E(x)) \geq \delta.$$

The right-hand inequality will be derived from (essentially) known results as well. It follows from duality theorems for $H^1(\Omega, \omega)$ and $H^1_{\text{prob}}(\Omega, \omega)$. Subsequently we will only indicate how proofs in [P] have to be modified to give the desired results.

Definition 2.8a. The Greens measure of Y_t with respect to Ω at x: $G(x)$ is defined by

$$G(x, A) := \int_\Sigma \int_0^{\tau(\xi)} \chi_A(y_s)\, ds\, d\mathbf{P}^x(\xi).$$

Remark. Let $\mu_{x,t}(A) := \mathbf{P}^x\{y_t \in A, t < \tau\}$ then $G(x, A) = \int_0^\infty \mu_{x,t}(A)\, dt$ (cf. [Øk, p. 140]).

Definition 2.9b. For $x \in \Omega$, $y \in \partial\Omega$ we let for $n > 2$

$$g(x, y) := e_n \left(|x - y|^{2-n} - \int_{\partial\Omega} |z - y|^{2-n}\, dw^x(z) \right).$$

$g(x, y)$ is called the Greens function of Ω. For $n = 2$, $|x - y|^{2-n}$ and $|z - y|^{2-n}$ are substituted by $\log|x - y|$ and $\log|z - y|$.

Remark. The connection between Greens measure and Greens function is given by

$$G(x, A) = \int_A g(x, y)\, dy.$$

Next we recall some

Identites 2.10. Let u, v be L-harmonic in Ω. Then for $x \in \Omega$:

$$(10.1)\qquad \int_\Omega \left(\sum_{i,j}^n a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\right)(y) g(x,y)\,dy = \int_{\partial\Omega} (u(y)-u(x))(v(y)-v(x))\,dw^x(y),$$

$$(10.2)\qquad \int_\Omega \left(\sum_{ij} a_{ij} \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_j}\right)(y) g(x,y)\,dy = \int_\Sigma \int_0^{\tau(\xi)} \left(\sum a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\right)(y_t(\xi))\,dt\,d\mathbf{P}^x(\xi),$$

$$(10.3)\qquad \mathbf{E}^x((u(y_\tau)-u(y_t))^2|y_t) = e_n \cdot \mathbf{E}^x\left(\int_t^\tau \left(\sum a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}(y_s)\,ds\right| y_t\right).$$

Remark. (10.1) is contained in [D-J-K, identity (7)]. (10.2) is a tautology using the connection Green measures and Green functions. (10.3) follows from (10.1), (10.2) as explained in [P, p. 86].

We will use identity (10.3) when necessary, otherwise we will use the argument of [P] as described there on pp. 89–91, to prove the following:

Lemma 2.11. *Let u be a L-harmonic in* Ω *and let*

$$(S_t(u))(\xi) := \left(\int_0^{t\wedge\tau(\xi)} \left(\sum a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}(y_s)\,ds\right)\right)^{1/2},$$

$$S_\infty(u)(\xi) := \lim_{t\to\infty} (S_t(u))(\xi).$$

Then there exists $C > 0$ *such that*

$$\int_\Sigma S_\infty(u)\,d\mathbf{P} \le C \int u^*\,d\mathbf{P}.$$

Lemma 2.12. *Let u, v be L-harmonic functions in* Ω *with boundary values* $\overline{u}$, $\overline{v}$ *such that* $u(0) = v(0) = 0$. *Then*

$$\int_{\partial\Omega} \overline{u}(x)\overline{v}(x)\,dw \le C \cdot \int_\Sigma S_\infty(u) d\mathbf{P} \cdot \left\| E\left(\int_t^\tau \left(\sum a_{ij} \frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\right)(y_s)\,ds \right| y_t\right)\right\|_\infty^{1/2}.$$

Proof. We assume $u(0) = v(0) = 0$. Then by (10.1), (10.2),

$$\begin{aligned}
\int_{\partial\Omega} u(x)v(x)\,dw(x) &= \int_\Sigma \int_0^\tau \left(\sum a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\right)(y_t)\,dt\,d\mathbf{P} \\
&\le \int_\Sigma \int_0^\tau \left(\sum a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right)^{1/2} \left(\sum a_{ij} \frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_i}\right)^{1/2} (y_t)\,dt\,d\mathbf{P} \\
&\le \left(\int_\Sigma \int_0^\tau \left(\sum a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right)(y_t) S_t^{-1}(u)\,dt\,d\mathbf{P}\right)^{1/2} \\
&\quad \cdot \left(\int_\Sigma \int_0^\tau \left(\sum a_{ij} \frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\right)(y_t) S_t(u)\,dt\,d\mathbf{P}\right)^{1/2}.
\end{aligned}$$

Following [P] we observe that the first factor equals

$$\int_{\Sigma}\int_0^{\tau} \frac{\frac{d}{dt}(S_t^2(u))}{S_t(u)}\, dt\, d\mathbf{P}$$

and the second factor equals

$$\int_{\Sigma}\int_0^{\tau} \left(\sum_{i,j=1}^{n} a_{ij}\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial u_j}\right)(y_t)S_t(u)\, dt\, d\mathbf{P}.$$

Using identity (10.3) we may treat both factors as in [P, pp. 94–95].

The argument needed for a proof of the next proposition is given by Jerison and Kenig in [J-K, Lemma 4.14 and Lemma 9.7] and will not be repeated here:

Proposition 2.13. *Let* $f \in L^1(\partial\Omega, \omega)$. *Fix* $x \in \Omega$, $f(x) := \int_{\partial\Omega} f(Q)\, dw^x(Q)$. *Then*

$$\sup_{x\in\Omega}\int_{\partial\Omega} |f(Q) - f(x)|^2\, dw^x(Q) \le \|f\|^2_{BMO(\partial\Omega,\omega)}.$$

Lemma 2.14. *Let* u *be* L*-harmonic in* Ω *with boundary values* u *in* $\partial\Omega$. *Then for* $t \ge 0$

$$\left\|E\left(\int_t^{\tau}\left(\sum a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right)(y_s)\, ds\middle| y_t\right)\right\|_{\infty} \le C\|u\|^2_{BMO}.$$

Proof. By (10.3) we may consider

$$\|E(u(y_\tau)^2 - u(y_t)^2|y_t)\|_{\infty} =: K.$$

$K < C\|u\|_{BMO}$ if for each $G \subset \Omega$, $t \in \mathbf{R}$,

$$\frac{1}{\mu_t(G)}\int_{y_t^{-1}(G)} E(|u(y_\tau) - u(y_t)|^2|y_t)\, d\mathbf{P} \le C\|u\|^2_{BMO}$$

which holds iff

$$\frac{1}{\mu_t(G)}\int_G\left(\int_{\Sigma}|u(y_\tau) - u(z)|^2\, d\mathbf{P}^z(\xi)\right) d\mu_t(z) \le C\|u\|^2_{BMO}.$$

This is implied by the following inequality

$$\sup_{t\in\Omega}\int_{\partial\Omega}|u(x) - u(z)|^2\, dw^z \le C\|u\|^2_{BMO}$$

which is true by Proposition 2.13.

Proof of Theorem 2.5 (right-hand side). Let U be a L-harmonic function in Ω. By the Duality Theorem 2.3 [J-K], there exists v, L-harmonic such that $\|v\|_{BMO(\partial\Omega,\omega)} = 1$ and

$$\frac{1}{C}\|u\|_{H^1(\Omega,\omega)} = \int_{\partial} \Omega u(x)v(x)\, dw(x),$$

where C is independent of u. Moreover, by Lemmas 2.12 and 2.14

$$\begin{aligned}\int_{\partial\Omega} u(x)v(x)\,dw(x) &\le \int_\Sigma u^*(\xi)d\mathbf{P}(\xi)\left(\sup_{y\in\Omega}\int_{\partial\Omega}(v(x)-v(y))^2\,d\omega^y(x)\right)^{1/2}\\ &\le \int_\Sigma u^*(\xi)d\mathbf{P}(\xi)\cdot C'\end{aligned}$$

where C' is independent of v and u. Hence there exists $\widetilde{C}\in\mathbf{R}^+$ such that

$$\|u\|_{H^1(\Omega,\omega)} \le \widetilde{C}\cdot\|u\|_{H^1_{\text{prob}}(\Omega,\omega)}.$$

When combined with [J-K, Theorem 8.14], Theorem 2.5, asserts that any statement about the isomorphic structure of $H^1_{\text{prob}}(\Omega,\omega)$ implies the corresponding statement about the isomorphic structure of $H^1(\partial\Omega,d,\omega)$.

To apply Maurey's probabilistic methods we still need the results about the regularity of L-harmonic functions. These will be derived from properties of the kernel function to be defined now.

First, it is well known that the measures ω^x are mutually absolutely continuous with respect to each other. Let $K(x,a)$ denote the Radon-Nikodym derivative of ω^x with respect to ω at $Q\in\partial\Omega$ (i.e. $K(x,Q)=d\omega^2(Q)/d\omega$.

Theorem 2.15 [C-F-M-S]. *The map* $u\colon x\to K(x,Q)$ *satisfies* $Lu=0$ *in* Ω.

Lemma 2.16. *For each* $u\in H^1(\Omega,\omega)$, $\varepsilon>0$ *and* $r>0$, *there exists* $\delta>0$ *such that for each* $x\in\Omega$ *with* $\operatorname{dist}(x,\complement\Omega)>r$ *we get*

$$\sup_{|x-y|<\delta}|u(x)-u(y)|<\varepsilon\|u\|_{H^1(\Omega,\omega)}.$$

Proof. We first show that there exists $\delta'>0$ and $C>0$, (depending only on r) such that for each $x\in\Omega$,

$$\sup_Q\int_{|x-y|<\delta'}K^2(y,Q)<C.$$

Indeed, by [J-K, Lemma 1.11], for $y\in\Omega$

$$K(y,Q)\le\frac{M}{\omega(\Delta(y_0,s))}$$

where $s=\operatorname{dist}(y,\partial\Omega)$ and $y_0\in\partial\Omega$ satisfies $|y-y_0|=s$. By Theorem 2.6 and Harnack's inequality for positive L-harmonic functions, for each $s>0$ there exists $\eta_s>0$ such that for each $x\in\partial\Omega$, we get $\omega(\Delta(z,s))>\eta_s$. Putting $\delta'=\frac{r}{4}$ we obtain

$$\sup_{Q\in\partial\Omega}\int_{|x-y|<\delta'}K^2(y,Q)\,dy\le\eta_{r/4}^{-2}M^2.$$

Now we recall that K is L-harmonic as a function of y. By Di Giorgi's theorem (cf. [G-T, Theorem 8.24]), which dominates the α-Lipschitz constant of an L-harmonic function by its L^2-norm, we obtain $C > 0$ such that for each $Q \in \partial\Omega$ and $x, y \in \Omega$

$$|K(x, Q) - K(y, Q)| \leq \eta_{r/4}^{-2}|x - y|^\alpha$$

where α depends on r. Now choose $\delta > 0$ such that $\eta_{r/4}^{-2}\delta^\alpha < \varepsilon$ and estimate

$$\begin{aligned}|u(x) - u(y)| &\leq \int_{\partial\Omega}(K(x, Q) - K(y, Q))u(Q)\,dw(Q)\\ &\leq \sup_{Q\in\partial\Omega}|K(x, Q) - K(y, Q)|\,||u||_{L(\partial\Omega, \omega)}\\ &\leq \varepsilon||u||_{H^1(\Omega, \omega)}\end{aligned}$$

provided $|x - y| < \delta$.

If we now feed the probabilistic machinery of B. Maurey [Ma] (see also Wolniewicz) [W] with Proposition 2.13 and Lemma 2.16, we obtain immediately

Theorem 2.20. *The Hardy space* $H^1_{\text{prob}}(\Omega, \omega)$ *is isomorphic to a complemented subspace dyadic* $H^1(\delta)$.

Hence by Theorem 2.10, Theorem 1.4 and the Banach space decomposition principle of Pelczynski we arrive at

Theorem 2.21. *The Hardy spaces* $H^1(\Omega, \omega)$, $H^1_{\text{prob}}(\Omega, \omega)$ *and* $H^1(\partial\Omega, d, \omega)$ *are isomorphic to dyadic* $H^1(\delta)$.

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